

Transversely Hessian foliations and information geometry

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Abstract

A family of probability distributions parametrized by an open domain Λ in R^n defines the Fisher information matrix on this domain which is positive semi-definite. In information geometry the standard assumption has been that the Fisher information matrix tensor is positive definite defining in this way a Riemannian metric on Λ . If we replace the "positive definite" assumption by the existence of a suitable torsion-free connection, a foliation with a transversely Hessian structure appears naturally. In the paper we develop the study of transversely Hessian foliations in view of applications in information geometry.

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1 Introduction – Information Geometry

The Fisher metric is one of the basic tools in information geometry. It is defined on an open domain in R^m which parametrizes the set of probability distributions under consideration. The assumptions made lead to the study of affine manifolds with Riemannian metrics, with both geometrical structures loosely related. Using the probability distributions one can define a $(0, 2)$ tensor field on the open domain. The assumption that the defined tensor field is positive definite is rather strong, therefore we propose to study the consequences of a weaker condition. In this situation a foliation with a very particular transverse structure appears. Such foliated manifolds are natural generalizations of Hessian manifolds.

Let Λ be a domain in R^m . We consider families of probability distributions on a set \mathcal{X} parametrized by $\lambda \in \Lambda$.

$$\mathcal{P} = \{p(x; \lambda) | \lambda \in \Lambda\}$$

- (1) Λ is a domain in R^m ,
- (2) $p(x; \lambda)$ for a fixed x is a smooth function in λ ,
- (3) the operation of integration with respect to x and differentiation with respect to λ are commutative.

Definition Let $\mathcal{P} = \{p(x; \lambda) | \lambda \in \Lambda\}$ be a family of probability distributions on a set \mathcal{X} parametrized by $\lambda \in \Lambda$. We set $l_\lambda = l(x; \lambda) = \log p(x; \lambda)$ and denote by E_λ the expectation with respect to $p_\lambda = p(x; \lambda)$. Then the matrix $g_F(\lambda) = [g_{ij}(\lambda)]$ defined by

$$g_{ij}(\lambda) = E_\lambda \left[\frac{\partial l_\lambda}{\partial \lambda^i} \frac{\partial l_\lambda}{\partial \lambda^j} \right] = \int_{\mathcal{X}} \frac{\partial l(x; \lambda)}{\partial \lambda^i} \frac{\partial l(x; \lambda)}{\partial \lambda^j} p(x; \lambda) dx$$

is called the Fisher information matrix tensor.

Simple calculations show, see [13], that

$$g_{ij}(\lambda) = -E_\lambda \left[\frac{\partial^2 l_\lambda}{\partial \lambda^i \partial \lambda^j} \right].$$

The Fisher information matrix tensor $g_F(\lambda) = [g_{ij}(\lambda)]$ is positive semi-definite on Λ :

$$\sum_{i,j} g_{ij}(\lambda) c^i c^j = \int_{\mathcal{X}} \left\{ \sum_i c^i \frac{\partial l(x; \lambda)}{\partial \lambda^i} \right\}^2 p(x, \lambda) dx \geq 0.$$

In information geometry the standard assumption has been, cf. [13], p.105,:

- (4) For a family of probability distributions $\mathcal{P} = \{p(x; \lambda) | \lambda \in \Lambda\}$ the Fisher information matrix tensor $g_F(\lambda) = [g_{ij}(\lambda)]$ is **positive definite** on Λ .

We **weaken this condition** assuming only that the Fisher information matrix tensor is a tensor field parallel with respect to some torsion-free connection on Λ . Then a foliation appears in a very natural way, and under some mild assumptions it has a transverse Hessian structure. The main part of this note is devoted to the development of the foundations of the theory of transversely Hessian foliation which can be applied to a classification of spaces of probability distributions in the non-regular case.

2 Foliations

Let \mathcal{F} be a foliation on an m -manifold M . Then \mathcal{F} is defined by a cocycle $\mathcal{U} = \{U_i, f_i, k_{ij}\}_{i \in I}$ modeled on a q -manifold N_0 ($0 < q < m$) such that

- (1) $\{U_i\}_{i \in I}$ is an open covering of M ,
- (2) $f_i : U_i \rightarrow N_0$ are submersions with connected fibres,
- (3) $k_{ij} : N_0 \rightarrow N_0$ are local diffeomorphisms of N_0 with $f_i = k_{ij} f_j$ on $U_i \cap U_j$.

The connected components of the trace of any leaf of \mathcal{F} on U_i consists of fibres of f_i . The open subsets $N_i = f_i(U_i) \subset N_0$ form a q -dimensional manifold $N_{\mathcal{U}} = \bigsqcup N_i$, which can be considered to be a complete transverse manifold of the foliation \mathcal{F} . The pseudogroup $\mathcal{H}_{\mathcal{U}}$ of local diffeomorphisms of N generated by k_{ij} is called the holonomy pseudogroup of the foliated manifold (M, \mathcal{F}) defined by the cocycle \mathcal{U} . The equivalence class \mathcal{H} of $\mathcal{H}_{\mathcal{U}}$, for the notion of the pseudogroup equivalence see [8, 9, 10], is called the holonomy group of \mathcal{F} , or of the foliated manifold (M, \mathcal{F}) . A foliation on a smooth manifold M understood as an involutive subbundle of TM , or equivalently, according to the Frobenius theorem, cf. [4], p.37, as a partition of the manifold by submanifolds of the same dimension with some regularity condition, can be defined by many different cocycles. There is a notion of equivalent cocycle, modeled on the notion of equivalent atlases of a smooth manifold, and a foliation can be understood as an equivalence class of such cocycles, cf., the notion of a smooth structure on a topological space. Moreover, a pseudogroup equivalent to a holonomy pseudogroup representative is itself a pseudogroup associated to some cocycle defining the foliation. Therefore in some cases, for our foliation, we will be able to choose a cocycle modeled on a particular manifold, cf. [16].

The vector bundle $N(M, \mathcal{F}) = TM/T\mathcal{F}$ is called the normal bundle of the foliation \mathcal{F} . Then the tangent bundle TM is isomorphic to the direct sum $T\mathcal{F} \oplus N(M, \mathcal{F})$. These isomorphisms are determined by the choice of a supplementary subbundle Q in TM to the tangent bundle to the foliation $T\mathcal{F}$. The cocycle $\mathcal{U} = \{U_i, f_i, k_{ij}\}_{i \in I}$ modeled on a q -manifold N_0 induces on the normal bundle a cocycle $\mathcal{V} = \{V_i, \bar{f}_i, \bar{k}_{ij}\}_{i \in I}$ modeled on the $2q$ -manifold TN_0 , where $V_i = TU_i$, \bar{f}_i is the mapping induced by df_i , and $\bar{k}_{ij} = dk_{ij}$. The foliation \mathcal{F}_N of the normal bundle is of codimension $2q$, its leaves project on leaves of \mathcal{F} . They are, in fact, coverings of these leaves. In a similar way one can foliate any bundle obtained via a point-wise process from the normal bundle, e.g., the frame bundle of the normal bundle, the dual normal bundle, any tensor product of these bundles.

Let $\phi : U \rightarrow R^p \times R^q$, $\phi = (\phi^1, \phi^2) = (x^1, \dots, x^p, y^1, \dots, y^q)$ be an adapted chart on a foliated manifold (M, \mathcal{F}) . Then on U the vector fields $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}$ span the bundle $T\mathcal{F}$ tangent to the leaves of the foliation \mathcal{F} , the equivalence classes of $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^q}$ denoted by $\frac{\bar{\partial}}{\partial y^1}, \dots, \frac{\bar{\partial}}{\partial y^q}$, span the normal bundle $N(M, \mathcal{F})$, in fact, these vector fields are foliated sections of the normal bundle foliated by the foliation \mathcal{F}_N .

In the case of a foliated manifold we can consider three types of geometrical structures related to the foliation:

transverse - defined on the transverse manifold, the associated holonomy pseudogroup consists of automorphisms of this geometrical structure;

foliated - only defined on the normal bundle, and when expressed in a local adapted chart, depending only on the transverse coordinates; a foliated structure projects to a transverse structure along submersions of the cocycle defining the foliation;

associated - defined globally, on the tangent bundle but adapted to the splitting, and defining a foliated structure on the normal bundle.

Foliated and transverse structures are in one-to-one correspondence, an associated structure defines a foliated structure, but different associated structures can define the same foliated structure, cf. [16].

Example 1 Let us see what these types of structures give in the case of a Riemannian metric on a foliated manifold (M, \mathcal{F}) with the foliation defined by a cocycle \mathcal{U} . A transverse Riemannian metric is a Riemannian metric \hat{g} on the transverse manifold N of which elements of the holonomy pseudogroup are local isometries. If it is true for one holonomy group representative, it is true for any equivalent pseudogroup. Such a Riemannian metric can be "lifted" to a metric tensor field g on the normal bundle. Locally, using the sections $\frac{\bar{\partial}}{\partial y^1}, \dots, \frac{\bar{\partial}}{\partial y^q}$ and their duals dy_1, \dots, dy_q such a tensor field can be written as

$$g = \sum_{i=1}^q g_{ij}(y) dy_i dy_j$$

Mind that the condition "foliated" is equivalent to the fact that the functions g_{ij} depend only on the variables y^1, \dots, y^q . Of course, a metric tensor on the normal bundle can be extended to a metric tensor on the tangent bundle TM using the splitting and the isomorphism we have discussed above. The normal bundle is isomorphic to any complementary subbundle Q of the tangent bundle $T\mathcal{F}$. Therefore the tensor field g can be transported to a metric tensor field g_Q on the subbundle Q . Making Q orthogonal to $T\mathcal{F}$ and choosing a metric tensor in $T\mathcal{F}$ we get a Riemannian metric \bar{g} on M , which induces a metric tensor g in the normal bundle. Such Riemannian metrics are called *bundle-like* and have very interesting properties and characterisations, cf. [12].

Example 2 Likewise there three types of connections "adapted" to a foliation. Let \mathcal{U} be a cocycle defining the foliation \mathcal{F} , $N_{\mathcal{U}}$ and $\mathcal{H}_{\mathcal{U}}$ the associated transverse manifold and holonomy pseudogroup, respectively. First, by a transverse connection we understand a connection on the transverse manifold $N_{\mathcal{U}}$ of which elements of the holonomy pseudogroup $\mathcal{H}_{\mathcal{U}}$ are affine transformations. If such a connection exists for one transverse manifold, it exists on any other transverse manifold. A transverse connection $\hat{\nabla}$ defines a foliated connection ∇ in the normal bundle by the formula

$$\bar{f}_i(\nabla_X \bar{Y}) = \hat{\nabla}_{\bar{f}_i(X)} \bar{f}_i(\bar{Y})$$

for any vector $X \in TU_i$ and any foliated section \bar{Y} on U_i . Conversely, by the same formula, any foliated connection defines a connection on the transverse manifold of which the holonomy pseudogroup consists of affine transformations.

Using the splitting of the tangent bundle TM we can extend any connection in the normal bundle to a connection in TM for which the subbundle $T\mathcal{F}$ is parallel. Conversely, any connection

$'\nabla$ for which $T\mathcal{F}$ is parallel defines a connection ∇ in the normal bundle by the formula below, where $\bar{\cdot}$ represents passing to the normal bundle (section):

$$\nabla_X \bar{Y} = ' \nabla_X \bar{Y}$$

for any vector fields X, Y . The induced connection ∇ is foliated if for any infinitesimal automorphism (i.a.) of the foliation X , and any foliated section \bar{Y} of the normal bundle, $\nabla_X \bar{Y}$ is a foliated section of the normal bundle, which is equivalent to the fact that for any infinitesimal automorphism Y of the foliation \mathcal{F} , $'\nabla_X Y$ is an infinitesimal automorphism of \mathcal{F} .

Example 3 A foliation \mathcal{F} is called transversely affine if there exists a flat foliated connection. This is equivalent to the existence of a transverse flat connection. That is the transverse manifold is an affine (flat) manifold. Affine manifolds are locally affinely isomorphic to open subsets of R^q with the standard flat connection. Therefore there exists a cocycle defining the foliation \mathcal{F} modelled on the R^q such that elements of the associated holonomy pseudogroup are restrictions of affine transformations of R^q . This is equivalent to the existence of an atlas adapted to the foliation such that the changes of the transverse coordinates are restrictions of affine transformations of R^q . Because that property transversely affine foliations are developable, cf. [17].

Remark Following [16], we will use the convention: "normal" when qualifying a geometrical object means that this object is defined only on the normal bundle. If such an object projects along local submersions defining the foliation, it will be called foliated. Holonomy invariant objects on a transverse manifold will be called transverse. So foliated objects are normal but not all normal objects are foliated. However, a connection ∇ in the normal bundle $N(M, \mathcal{F})$ will be called normal if $\nabla_X \bar{Y} = 0$ for any foliated section \bar{Y} and $X \in T\mathcal{F}$. This apparently inconsistency is only superficial as only with that condition a connection in the normal bundle can be defined as a section of a suitable associated bundle derived from the normal bundle.

3 Transversely Hessian foliations

Let us continue with the study of a family of probability distributions as described in Introduction. Assume that there exists a torsion-free connection $'\nabla$, cf. [11], for which the tensor g_F is parallel, i.e., $'\nabla g_F = 0$, and define the distribution $ker g_F$:

$$ker g_F = \{v \in TM : g_F(v, v) = 0\} = \{v \in TM : g_F(v, w) = 0 \quad \forall w \in TM\}$$

$ker g_F$: is parallel with respect to $'\nabla$ as

$$0 = ' \nabla_X g_F(Y, Z) = X g_F(Y, Z) - g_F(' \nabla_X Y, Z) - g_F(Y, ' \nabla_X Z)$$

where $X, Y, Z \in TM$. If $Y \in \text{ker}g_F$, $g_F(Y, Z) = 0$ for any Z , and we get that $g_F(''\nabla_X Y, Z) = 0$ for any $Z \in TM$, which ensures that $''\nabla_X Y \in \text{ker}g_F$ provided that $Y \in \text{ker}g_F$. As the connection $''\nabla$ is torsion-free the distribution $\text{ker}g_F$ is involutive:

$$0 = T(X, Y) = ''\nabla_X Y - ''\nabla_Y X - [X, Y]$$

so

$$[X, Y] = ''\nabla_X Y - ''\nabla_Y X \in \text{ker}g_F$$

for any vector fields $X, Y \in \text{ker}g_F$. Moreover, the distribution $\text{ker}g_F$ is of constant dimension as $''\nabla_Y X \in \text{ker}g_F$ for any $X \in \text{ker}g_F$ and any vector field Y . This means that the parallel transport along any curve maps vectors from $\text{ker}g_F$ to vectors from $\text{ker}g_F$, which ensures that the distribution $\text{ker}g_F$ is of constant dimension, thus defines a foliation \mathcal{F} .

The tensor field g_F induces a (normal) Riemannian metric g in the normal bundle $N(M, \mathcal{F})$. The connection $''\nabla$ defines a connection ∇ in the normal bundle $N(M, \mathcal{F})$ which is the Levi-Civita connection of the metric g . Indeed

$$\nabla_X g(\bar{Y}, \bar{Z}) = Xg(\bar{Y}, \bar{Z}) - g(\nabla_X \bar{Y}, \bar{Z}) - g(\bar{Y}, \nabla_X \bar{Z}) =$$

$$Xg_F(Y, Z) - g(''\nabla_X Y, \bar{Z}) - g(\bar{Y}, ''\nabla_X Z) = Xg_F(Y, Z) - g_F(''\nabla_X Y, Z) - g_F(Y, ''\nabla_X Z) = 0$$

where X is any vector field, Y, Z are i.a.s of the foliation. So the connection ∇ is g -metric. Obviously, it is torsion-free, hence ∇ is the Levi-Civita connection of the Riemannian metric g of the normal bundle.

The induced metric will be foliated if $Xg(\bar{Y}, \bar{Z}) = Xg_F(Y, Z) = 0$ for any vector field X tangent to the foliation \mathcal{F} and any foliated vector fields Y and Z . It is the case if $''\nabla_X Y \in T\mathcal{F}$ for any vector field X tangent to the foliation \mathcal{F} and any foliated vector fields Y . As the connection $''\nabla$ is torsion-free this condition is equivalent to $''\nabla_Y X \in T\mathcal{F}$ for any vector $Y \in TM$ and any vector field $X \in T\mathcal{F}$.

The flat connection $'D$ of Λ should be related in some way to the foliation \mathcal{F} . If we assume a similar condition for $'D$, i.e. $'D_Z W \in T\mathcal{F}$ for any vector $Z \in TM$ and any vector field $W \in T\mathcal{F}$, then we can define a connection D in the normal bundle by the formula

$$D_X \bar{Y} = 'D_X \bar{Y}$$

where \bar{Y} denotes the section of the normal bundle defined by the vector field Y . The normal connection D is flat. The connection D is transversally projectable (foliated) if $D_X s$ is a foliated section for any foliated section s and a foliated vector field X , i.e. an infinitesimal automorphism of \mathcal{F} .

Let \mathcal{F} be a foliation of codimension q on a manifold M of dimension m . The dimension of its leaves is p , i.e. $p + q = m$. Assume that the foliation \mathcal{F} is transversely affine, cf. [7, 17, 18]. A Riemannian metric \hat{g} is bundle-like if for any adapted chart $\varphi = (x^1, \dots, x^p, y^1, \dots, y^q)$, $\hat{g} = \sum_{i,j=1}^p g_{ij}(x, y)v^i v^j + \sum_{a,b=1}^q g_{ab}(y)dy^a dy^b$, where v^i is the only 1-form which vanishes on the orthogonal complement of the bundle $T\mathcal{F}$ and $v^i(\frac{\partial}{\partial x^j}) = \delta_j^i$. A bundle-like metric \hat{g} is said to be transversely Hessian if the horizontal part g of the metric is expressed by the formula

$$g = \sum_{i,j=1}^q \frac{\partial^2 h}{\partial y^i \partial y^j} dy^i dy^j$$

Remark In principle h need not be basic, i.e. it may depend on variables x^i , we just assume that it does not.

Definition We say that the foliation is transversely Hessian if

- it is transversely affine
- it admits a bundle-like metric which is transversely Hessian.

Therefore a transversely Hessian foliation is at the same time a Riemannian foliation and a transversely affine foliation although both structures may have not much in common.

A foliation \mathcal{F} is transversely Hessian iff for some cocycle \mathcal{U} defining the foliation the associated transverse manifold $N_{\mathcal{U}}$ admits a Hessian structure of which elements of the holonomy pseudogroup are automorphisms, i.e. local affine transformations of the flat connection and isometries of the Hessian metric.

Let ∇ be the (normal) Levi-Civita connection of the foliated Riemannian metric g in $N(M, \mathcal{F})$. ∇ is a foliated connection. The difference $\gamma = \nabla - D$ is also a normal tensor which is foliated iff D is a foliated connection. Moreover, as both connections are torsion-free, $\gamma_X Y = \gamma_Y X$.

Like in the standard case we have the following proposition

Theorem 1. Let (M, \mathcal{F}) be a foliated manifold. Assume that \mathcal{F} is transversely affine with foliated flat connection D and g is a foliated metric on (M, \mathcal{F}) . Then, in any adapted chart (U, ϕ) the following conditions are equivalent, where i, j, k run from 1 to q :

- i) g is a foliated Hessian metric;
- ii) $D_X g(Y, Z) = D_Y g(X, Z)$ for any foliated sections X, Y, Z of the normal bundle;
- iii) $\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{kj}}{\partial y^i}$;
- iv) $g(\gamma_X Y, Z) = g(Y, \gamma_X Z)$ for any foliated sections X, Y, Z of the normal bundle;
- v) $\gamma_{ijk} = \gamma_{jik}$.

The proof follows closely that Proposition 2.1 of [13]. First of all, notice that as the metric g foliated, (i) implies (iii) from the very definition of a Hessian metric. Let us choose and adapted atlas such that the transverse coordinate changes are affine transformations. Then the sections

$\frac{\partial}{\partial y_i}$ are D -parallel. In such a coordinate system (iii) is just a local expression of (ii) and (v) of (iv).

Moreover, γ_{ijk} are related to the Christoffel symbols of the foliated Levi-Civita connection ∇ . As such we have the following expression for them

$$\Gamma_{jk}^i = g^{is} \left(\frac{\partial g_{sj}}{\partial y_k} + \frac{\partial g_{sk}}{\partial y_j} - \frac{\partial g_{jk}}{\partial y_s} \right)$$

and

$$\gamma_{ijk} = g_{is} \Gamma_{jk}^s = \left(\frac{\partial g_{ij}}{\partial y_k} + \frac{\partial g_{ik}}{\partial y_j} - \frac{\partial g_{jk}}{\partial y_i} \right)$$

the derivatives with respect to variables along leaves of the foliation not appearing as the connection is foliated. It is clear that the conditions (iii) and (v) are equivalent.

Let us demonstrate that (iii) implies (i). As the metric g and the connection D are foliated, and the local coordinate system is adapted to the transversely affine foliation, the implication is equivalent to the corresponding fact for (transverse) manifolds. But it was proved in Proposition 2.1 of [13].

Let \mathcal{F} be a transversely affine foliation on a manifold M , let $N(M, \mathcal{F})$ be its normal bundle. It admits a foliation \mathcal{F}_N of the same dimension as \mathcal{F} but of codimension $2q$.

Let $\varphi = (x^1, \dots, x^p, y^1, \dots, y^q)$ be an adapted chart. Then on the normal bundle $N(M, \mathcal{F})$ we have an adapted chart $\varphi^N = (x^1, \dots, x^p, y^1, \dots, y^q, dy^1, \dots, dy^q)$, of course the coordinates x^i are identified with $x^i|_p$ where p is the projection onto the base M in the normal bundle. Moreover, putting $z^i = y^i + \sqrt{-1}dy^i$ we define transversely holomorphic coordinate system on the normal bundle. Therefore if the foliations \mathcal{F} is transversely affine the foliation \mathcal{F}_N is transversely holomorphic for a foliated complex structure J_N . Moreover, on the normal bundle we define a normal Riemannian metric g^N by the formula locally expressed

$$g^N = \sum_{ij}^q g_{ij} p d\bar{z}^i d\bar{z}^j$$

In this case we have the following proposition

Proposition 1. *Let (M, \mathcal{F}) be a foliated manifold, and g be a foliated metric. Then the following conditions are equivalent:*

- (1) g is a foliated Hessian metric on (M, \mathcal{F}, D)
- (2) g^N is a foliated Kählerian metric on $(N(M, \mathcal{F}), \mathcal{F}_N, J_N)$.

From the very formulae both metrics are foliated. Both properties are local, so can be considered in a suitable adapted chart. In that case property (1) is equivalent to the fact that the induced transverse metric g^T on $N_{\mathcal{U}}$ is Hessian, and property (2) to the fact that the induced metric g^{NT} on the tangent bundle of the manifold $N_{\mathcal{U}}$ is Kählerian. But it is precisely the substance of Proposition 2.6 of [13].

4 Dual foliated connections

Let (M, \mathcal{F}, D, g) be a transversely Hessian foliated manifold. Let ∇ be the Levi-Civita connection in the normal bundle of \mathcal{F} associated to the foliated Riemannian metric g . Then the connection

$$D' = 2\nabla - D$$

is a flat foliated connection in the normal bundle and

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D'_X Z).$$

Moreover, (M, \mathcal{F}, D', g) is a transversely Hessian foliated manifold. D' is called the dual connection of the transversely Hessian foliated manifold (M, \mathcal{F}, D, g) .

Indeed, let us choose a cocycle \mathcal{U} defining the foliation. The associated transverse manifold $N_{\mathcal{U}}$ is a Hessian manifold of which the associated holonomy pseudogroup $\mathcal{H}_{\mathcal{U}}$ is a pseudogroup of local automorphisms. Both connections D and ∇ are foliated, so they correspond to \bar{D} and $\bar{\nabla}$, respectively. We can apply Corollary 2.1 of [13] to this Hessian structure with connections $\bar{\nabla}$ and \bar{D} being the Levi-Civita connection and flat connection, respectively. The formula for the dual connection \bar{D}' (from Corollary 2.1 of [13]) assures that any local automorphism of the Hessian structure is an affine transformation of the connection \bar{D}' , so it is a transverse connection of our foliated structure. Therefore it defines a foliated connection D' for which we have been looking for.

This situation can be well described using the notion of a Codazzi pair of connections. We say that a Riemannian metric \bar{g} and a connection \bar{D} in the normal bundle are related by the normal Codazzi equations if

$$D_{\bar{X}}g(\bar{Y}, \bar{Z}) = D_{\bar{Y}}g(\bar{X}, \bar{Z})$$

for any foliated sections $\bar{X}, \bar{Y}, \bar{Z}$ of the normal bundle.

Definition *A pair (D, g) , g being a Riemannian metric in the normal bundle, and D a torsion-free connection in this vector bundle, is called a normal Codazzi structure if it satisfies the Codazzi equation for any i.a.s X, Y and any foliated section Z :*

$$D_Xg(\bar{Y}, Z) = D_Yg(\bar{X}, Z).$$

If both objects are foliated, then the pair (D, g) is called a *foliated Codazzi structure*. A foliated manifold (M, \mathcal{F}) is called a *foliated Codazzi manifold*, and is denoted $(M, \mathcal{F}; D, g)$. If g is a foliated Riemannian metric and D a foliated connection, then the pair is called a *foliated Codazzi manifold*.

Having a normal Codazzi structure (D, g) on a foliated manifold (M, \mathcal{F}) we can define a new normal torsion-free connection D' by the formula

$$Xg(\bar{Y}, \bar{Z}) = g(D_X\bar{Y}, \bar{Z}) + g(\bar{Y}, D'_X\bar{Z})$$

for any i.a.s X, Y, Z . The connection D' is called the dual connection of the connection D with respect to the Riemannian metric g .

Let us check that the connection D' is normal. If X is tangent to the foliation then $D_X\bar{Y} = 0$ as the connection D is normal. Moreover, as Y and Z are i.a.s $g(\bar{Y}, \bar{Z})$ is a basic function, so $Xg(\bar{Y}, \bar{Z}) = 0$. Thus $g(\bar{Y}, D'_X\bar{Z}) = 0$ for any i.a. Y , which ensures that in this case $D'_X\bar{Z} = 0$.

The connection \bar{D}' is torsion-free as

$$g(\bar{Y}, D'_X\bar{Z} - D'_Z\bar{X}) = g(\bar{Y}, D_X\bar{Z} - D_Z\bar{X})$$

for any i.a.s X, Y, Z . So D' is torsion-free iff the connection D is.

Moreover, we have the following proposition

Proposition 2. *On a foliated manifold (M, \mathcal{F}) , if the Codazzi structure (D, g) is foliated, so is the dual connection D' .*

If the Codazzi structure (D, g) is foliated, for any foliated sections X, Y, Z the functions $Xg(Y, Z)$ and $g(D_X Y, Z)$ are basic, so the function $g(Y, D'_X Z)$ is basic as well. That fact ensures that $D'_X Z$ is a foliated section for any foliated sections X, Z of the normal bundle, which means that the connection D' is foliated.

In the case of normal connections we have the following lemma

Lemma 1. *Let g be a foliated metric on the normal bundle. Let (D, D') be a pair of dual connections with respect to g . Then if one of the connections is foliated so is the other.*

The pair of connections satisfies the equation

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D'_X Z).$$

Let us check that D' is a foliated connection if D is. Let Y, Z be two foliated sections of $N(M, \mathcal{F})$.

If X is a vector field tangent to the leaves, then $Xg(Y, Z) = 0$ and $D_X Y = 0$, so $D'_X Y = 0$ and the connection D' is normal.

If X is an i.a. of \mathcal{F} , then the function $Xg(Y, Z)$ is foliated (basic). Likewise the function $g(D_X Y, Z)$ is foliated. Therefore $g(D'_X Y, Z)$ is a foliated function, and the connection D' is foliated.

If $(M, \mathcal{F}; D, g)$ is a foliated Codazzi manifold, so is $(M, \mathcal{F}; D', g)$. In that case the induced connections \bar{D} and \bar{D}' on the transverse manifold are dual with respect to the transverse Riemannian metric \bar{g} and the holonomy pseudogroup consists of affine transformations of both connections.

The following lemma is a foliated version of a lemma for connections on manifolds, cf. Lemma 2.3 of [13]. The proof is basically the same.

Lemma 2. *Let D be a torsion-free connection and let g be a Riemannian metric in the normal bundle. Define a new connection D' by*

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D'_X Z)$$

for any vector field X and any foliated sections Y, Z of the normal bundle. Then the following conditions (1)-(3) are equivalent:

- (1) the connection D' is torsion free,
- (2) the pair (D, g) satisfies the Codazzi equation

$$D_X g(Y, Z) = D_Y g(X, Z),$$

(3) let ∇ be the Levi-Civita connection for g , and let $\gamma_X Y = \nabla_X Y - D_X Y$. Then g and γ satisfy

$$g(\gamma_X Y, Z) = g(Y, \gamma_X Z).$$

If the pair (D, g) satisfies the Codazzi equation, then the pair (D', g) also satisfies this equation and

$$D' = 2\nabla - D,$$

$$D_X g(Y, Z) = 2g(\gamma_X Y, Z)$$

Remark Proposition 1 asserts that a flat connection D and a Riemannian metric g in the normal bundle of a foliation \mathcal{F} form a normal Hessian structure iff they satisfy the Codazzi equation. The same is true for foliated structures.

The curvature tensor R_D of a connection D in the normal bundle is defined as

$$R_D(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z$$

for vector fields X, Y and a section Z of the normal bundle. If the connection D is normal ($D_X Z = 0$ for any vector tangent to leaves) the tensor field R_D defines a tensor field denoted by the same letters R_D a section of the bundle

$$N(M, \mathcal{F})^* \otimes N(M, \mathcal{F})^* \otimes N(M, \mathcal{F})^* \otimes N(M, \mathcal{F})$$

This bundle is also foliated, cf. [16], and if the connection D is foliated so the section R_D . If the connection D is foliated the (foliated) curvature tensor field R_D corresponds to the curvature tensor field of the induced connection \bar{D} on the transverse manifold.

Therefore if the connection D is foliated we can introduce the following definition

Definition 1. *A foliated Codazzi structure is said to be of constant curvature c if the curvature tensor R_D of the connection D satisfies*

$$R_D(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y)$$

for any sections X, Y, Z of the normal bundle.

Remark The above condition is equivalent to the transverse Codazzi structure (\bar{g}, \bar{D}) being of constant curvature c as the curvature tensor R_D of the connection D projects, corresponds, to the curvature tensor $R_{\bar{D}}$ of the connection \bar{D} on the transverse manifold corresponding, induced by, the foliated connection D .

The propositions below are foliated versions of Propositions 2.8 and 2.9 of [13]. They are immediate consequences of the just mentioned Propositions applied to the induced Codazzi structures on the transvere manifold of the foliated manifold (M, \mathcal{F}) and of the above Remark.

Proposition 3. *Let (M, \mathcal{F}, D, g) be a foliated Codazzi manifold, and (D', g) be its dual Codazzi structure.*

(1) *Denoting by R_D and $R_{D'}$ the curvature tensors of D and D' , respectively, we have*

$$g(R_D(X, Y)Z, W) + g(Z, R_{D'}(X, Y)W) = 0,$$

for any sections X, Y, Z, W of the normal bundle.

(2) *If (D, g) is a Codazzi structure of constant curvature c , then (D', g) is also a Codazzi structure of constant curvature c .*

Proposition 4. *A foliated Codazzi structure (D, g) is of constant curvature 0 iff the foliation \mathcal{F} is transversely Hessian with the foliated structure given by the pair (D, g) .*

5 Normal curvatures

Let (D, g) be a normal Hessian structure on a foliated manifold (M, \mathcal{F}) . Let $\gamma = \nabla - D$ be the difference tensor of the normal Levi-Civita connection of g and D .

A normal tensor field Q of type (1,3) defined as

$$Q = D\gamma$$

is called the normal Hessian curvature tensor of the normal Hessian structure (D, g) .

If the structure is foliated so is its normal Hessian curvature tensor. The components Q_{jkl}^i of Q with respect to an adapted foliated affine coordinate system $(x^1, \dots, x^p, y^1, \dots, y^q)$ are given by, as the formula corresponds to the formula on the transverse manifold, see Chapter 3 of [13]:

$$Q_{jkl}^i = \frac{\partial \gamma_{jl}^i}{\partial y^k}.$$

The propositions below are foliated versions of Propositions 3.1 and 3.2 of [13]:

Proposition 5. *Let us consider an adapted foliated affine coordinate system $(x^1, \dots, x^p, y^1, \dots, y^q)$, and let $g_{ij} = \frac{\partial^2 \varphi}{\partial y^i \partial y^j}$ for some basic function φ . Then*

$$(1) \quad Q_{ijkl} = \frac{1}{2} \frac{\partial^4 \varphi}{\partial y^i \partial y^j \partial y^k \partial y^l} - \frac{1}{2} g^{rs} \frac{\partial^3 \varphi}{\partial y^i \partial y^k \partial y^r} \frac{\partial^3 \varphi}{\partial y^j \partial y^l \partial y^s}.$$

$$(2) \quad Q_{ijkl} = Q_{klji} = Q_{kjil} = Q_{ilkj} = Q_{jilk}.$$

for $i, j, k, l = 1, \dots, q$.

Proposition 6. *Let R be the Riemannian curvature tensor for the normal Riemannian metric g . Then*

$$R_{ijkl} = \frac{1}{2} (Q_{ijkl} - Q_{jikl}).$$

In Section **Transversely Hessian Foliations** we have demonstrated that when (M, \mathcal{F}) is transversely Hessian, the foliation \mathcal{F}^N of the normal bundle $N(M, \mathcal{F})$ is transversely Kähler for the lifted metric. The next proposition explains the relation between the tensor Q and the curvature tensor of this Kählerian metric.

Proposition 7. *Let (M, \mathcal{F}) be transversely Hessian. Let R^N be the Riemannian curvature tensor for the normal Kählerian metric g^N of the foliated manifold $(N(M, \mathcal{F}), \mathcal{F}^N)$. Then for any adapted foliated affine coordinate system*

$$R_{ijkl}^N = \frac{1}{2} Q_{ijkl} \circ \pi$$

where π is the natural projection $N(M, \mathcal{F}) \rightarrow M$.

It is a foliated version of Proposition 3.3 of [13], a simple consequence of the correspondence between "foliated" and "transverse" objects and of the fact that the foliation \mathcal{F}^N of the normal bundle is defined by the cocycle derived from a cocycle defining the foliation \mathcal{F} which is described at the beginning of section *Foliations*.

A normal metric tensor defines a q -form ω by the standard formula

$$\omega_x(v_1, \dots, v_q) = \det[g(v_i, v_j)]$$

where $v_1, \dots, v_q \in N(M, \mathcal{F})_x$. It can be understood as a section of $\wedge^q N(M, \mathcal{F})^*$. The form ω is called the normal volume form defined by g . If the metric g is foliated, the the form ω is foliated (basic). The corresponding q form $\bar{\omega}$ on the transverse manifold is the volume form defined by the transverse metric \bar{g} . We define a closed 1-form α and a symmetric bilinear normal form β by

$$D_X \omega = \alpha(X) \omega, \quad \beta = D\alpha.$$

The forms α and β are called the first normal Koszul form and the second normal Koszul form, respectively, of the normal Hessian structure (D, g) on a foliated manifold (M, \mathcal{F}) . In the case of a foliated structure both forms are basic (foliated). The next proposition is a foliated version of Proposition 3.4 of [13].

Proposition 8. *Let (M, \mathcal{F}) be a transversely Hessian foliated manifold. Then for any adapted foliated affine coordinate system, with indices going from 1 to q ,*

- (1) $\alpha(X) = \text{Tr}_N \gamma_X$,
- (2) $\alpha_i = \frac{1}{2} \frac{\partial \log \det[g_{kl}]}{\partial y^i} = \gamma_{ri}^r$
- (3) $\beta_{ij} = \frac{\partial \alpha_i}{\partial y^j} = \frac{1}{2} \frac{\partial^2 \log \det[g_{kl}]}{\partial x^i \partial y^j} = Q^r_{rij} = Q_{ij}^{r} r$.

The form β is related to the normal Ricci tensor \mathbf{R}^N of the normal Kählerian metric g^N of the foliated manifold $(N(M, \mathcal{F}), \mathcal{F}^N)$. The foliated version of Proposition 3.5 of [13] reads as follows:

Proposition 9. *Let (M, \mathcal{F}) be transversely Hessian. Then for any adapted foliated affine coordinate system $(x^1, \dots, x^p, y^1, \dots, y^q)$, let \mathbf{R}_{ij}^N be the local expression, with indeces going i, j from 1 to q , of be the normal Ricci tensor of the normal Kählerian metric g^N of the foliated manifold $(N(M, \mathcal{F}), \mathcal{F}^N)$. Then*

$$\mathbf{R}_{ij}^N = -\frac{1}{2} \beta_{ij} \circ \pi.$$

If the normal Hessian structure (D, g) satisfies the condition

$$\beta = \lambda g, \quad \lambda = \frac{\beta^i_i}{q}$$

then the normal Hessian structure is called Einstein-Hessian. The theorem below explains the above condition and its relation to the well-known Einstein-Kähler condition for Kählerian metrics. It is a direct consequence of the previous proposition.

Theorem 2. *Let (M, \mathcal{F}) be a foliated manifold, and (D, g) be a normal Hessian structure. Let (J^N, g^N) be the normal Kähler structure of the foliated manifold $(N(M, \mathcal{F}), \mathcal{F}^N)$ induced by (D, g) . Then the following conditions are equivalent*

- (1) (M, \mathcal{F}, D, g) is Einstein-Hessian;
- (2) $(N(M, \mathcal{F}), \mathcal{F}^N, J^N, g^N)$ is Einstein-Kählerian.

Normal Hessian sectional curvature

Let Q be the normal Hessian curvature tensor. The formula, $i, j, k, l = 1, \dots, q$,

$$\hat{Q}^{ik}(\xi) = \hat{Q}^i{}_j{}^k{}_l \xi^{jl}$$

defines an endomorphism \hat{Q} on the space of symmetric contravariant normal two tensors, i.e., on $\otimes^2 N(M, \mathcal{F})$. \hat{Q} is self-dual (symmetric) with respect to the scalar product tensor induced by the Hessian metric g .

Let us define a function q on the space $\otimes^2 N(M, \mathcal{F})$ by the formula

$$q(\xi) = \frac{\langle \hat{Q}(\xi), \xi \rangle}{\langle \xi, \xi \rangle}$$

for any $\xi \in \otimes^2 N(M, \mathcal{F})$ and $\langle \cdot, \cdot \rangle$ the inner product defined by the normal Riemannian metric g of the foliated Hessian structure. The function q is called the normal Hessian sectional curvature. It is a basic (foliated) function if the normal Hessian structure is foliated. The manifold $\otimes^2 N(M, \mathcal{F})$ admits the induced (natural) foliation of dimension p whose leaves are coverings of leaves of the foliation \mathcal{F} , [16].

We say that the foliated Hessian structure is of constant normal sectional curvature if q is a constant function on $\otimes^2 N(M, \mathcal{F})$. As the structures are foliated, and we have demonstrated that the discussed tensors are also foliated the results below, the foliated versions of Proposition 3.6, Corollary 3.1 and 3.2 of [13] are corollaries of these results when the correspondences between foliated and transverse geometric objects is applied.

Proposition 10. *The normal Hessian sectional curvature of a foliated Hessian structure (M, \mathcal{F}, D, g) is constant and equal to c iff for any adapted foliated affine coordinate system $(x^1, \dots, x^p, y^1, \dots, y^q)$,*

$$Q_{ijkl} = \frac{c}{2} \{g_{ij}g_{kl} + g_{il}g_{jk}\}$$

for any $i, j, k, l = 1, \dots, q$.

Corollary 3. *Let (M, \mathcal{F}, D, g) be a foliated Hessian structure. Then the following two conditions are equivalent:*

- (1) *the Hessian normal sectional curvature is a constant c ;*
- (2) *the holomorphic normal sectional curvature of $(N(M, \mathcal{F}), J^N, g^N; \mathcal{F}^N)$ is constant and equal to $-c$.*

Corollary 4. *Let (M, \mathcal{F}, D, g) be a foliated Hessian structure of constant normal Hessian sectional curvature c . Then the foliated Riemannian manifold (M, \mathcal{F}, g) is a foliated Riemannian manifold modeled on a space form of constant section curvature $-\frac{c}{4}$.*

6 Transversely Hessian foliations as developable foliations

Let X be a manifold and G a subgroup of $Diff(M)$ of diffeomorphisms of X . We say that a foliation \mathcal{F} of a smooth manifold M is a (X, G) -foliation if it admits a cocycle $\mathcal{U} = \{U_i, f_i, k_{ij}\}_{i \in I}$ modeled on X such that the mappings k_{ij} are restrictions of elements of the group G . Such a foliation is developable provided the action of G is quasi-analytic, cf. [7, 15], i.e., if two diffeomorphisms of G are equal on an open subset of X , then they are equal. For example it is true for isometries of a Riemannian manifold. In such a case there exist

i) a representation h of the fundamental group of M into G

$$h: \pi_1(M, x_0) \rightarrow G$$

ii) a developing mapping D of the universal covering \tilde{M} onto X , i.e. a submersion

$$D: \tilde{M} \rightarrow X$$

which is $\pi_1(M, x_0)$ -equivariant for the natural action on the universal covering space and the action on the manifold X via the representation h .

iii) the fibres of D define the foliation of \tilde{M} which projects onto the foliation \mathcal{F} , i.e. the foliation by the connected components of the fibres of D is the lift of the foliation \mathcal{F} to \tilde{M} .

Transversely Hessian foliations are developable as they are transversely affine, and as such they are $(R^q, Aff(R^q))$ -foliations. Therefore the universal covering of Λ admits a development D onto R^q with imD being an open subset R^q , which is invariant with respect to the action of $\pi_1(M, x_0)$ via the representation h .

Developable foliations have been studied in depth. It is easy to see that the transverse geometry can be read on imD , and elements of imh are automorphisms of these geometrical structures. So in the case transversely Hessian foliations on imD in addition to the obvious flat connections we have a Riemannian metric of whose elements of imh are isometries. In short, imD is a Hessian manifold and elements of imh are automorphisms of this Hessian structure.

There are two salient questions to be answered:

- a) is the developing mapping surjective?
- b) are the fibres of D diffeomorphic?

The first question is a question about transverse completeness of the foliation. There are no simple answers but for some see [18]. The second can be answered using the properties of the foliated Hessian metric. In fact, the developing mapping D is a Riemannian submersion, and if the foliated Hessian metric is transversely complete, then D is a locally trivial bundle. Local trivializations are obtained using geodesics orthogonal to the fibres. It is not easy to demonstrate transverse completeness of the metric. For example, it is the case when the foliated manifold is compact. Returning to the initial example of a family of probability distributions parametrized by

an open subset Λ of R^m it would be the case if Λ admitted a cocompact group K of automorphisms of our transversely Hessian foliation, i.e. the quotient topological space Λ/K is a compact manifold and the foliation \mathcal{F} projects to the transversely Hessian foliation \mathcal{F}_K of Λ/K . Another possibility is the geodesic completeness of the foliated affine structure, cf. [18], but even the existence of a cocompact group of automorphisms does not ensure that this structure is geodesic complete. The results of the paper mentioned above point to the importance of the properties of the group imh , called the holonomy group of the foliation. Its closure in $Aff(R^q)$ and its Zariski closure play a particularly important role, see also [7].

The existence of two related transverse structures, Riemannian and affine, permits to combine the results of two well-developed theories. For both types of foliations we have good estimates of the growth type of leaves:

- Riemannian foliations, see [5, 3]
- transversely affine, see [17, 18, 1]

The growth type of leaves is bounded from above by the growth type of the holonomy group.

The positive answer to the two questions formulated above definitely restricts the topology type of the manifold M , and the domain Λ of the principal example.

The local triviality of the developing mapping D ensures that the leaves of the foliation have diffeomorphic universal coverings provided that the manifold M is connected.

The existence of a transverse invariant measure is assured by the fact that the foliation is Riemannian, for results and properties of invariant measures for transversely affine foliations see [7], also see [17].

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